Optimization Problems in Portfolio Management

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Asset Pricing Theory

Time	t (today)	$t+1$ (tomorrow)
Price	p_t	
Payoff		x_{t+1}
Consumer	$c_t = e_t$	$c_{t+1} = e_{t+1}$
Investor	$c_t = e_t - N \cdot p_t$	$c_{t+1} = e_{t+1} + N \cdot x_{t+1}$
Utility	$u(c_t)$	$u(c_{t+1})$

Utility Optimization Problem

$$\mathcal{U} = u(c_t) + \beta\, \mathrm{E}_t[u(c_{t+1})] \longrightarrow \max$$

$$\frac{\partial \mathcal{U}}{\partial N} = 0 \quad \Longrightarrow \quad u'(c_t) \cdot (-p_t) + \beta \mathrm{E}_t[u'(c_{t+1}) x_{t+1}] = 0$$

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The Pricing Equation

$$\mathcal{U} \longrightarrow \max \quad \Longrightarrow \quad p_t = \mathrm{E}_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right]$$

$$p_t = \mathrm{E}_t[m_{t+1} x_{t+1}]$$

No uncertainty

$$x_{t+1} = p_t \cdot R_f \quad \Longrightarrow \quad p_t = \frac{1}{R_f} x_{t+1}$$

Risky asset

$$p_t^i = \frac{1}{R^i} \mathrm{E}_t[x_{t+1}^i]$$

Systematic Risk and Idiosyncratic Risk

Price = discounted present-value + risk adjustment

$$\left.\begin{aligned} p &= \mathrm{E}[mx] \\ \mathrm{Cov}(m,x) &= \mathrm{E}[mx] - \mathrm{E}[m]\mathrm{E}[x] \\ R_f &= \frac{1}{\mathrm{E}[m]} \end{aligned}\right\} \implies \quad p = \frac{\mathrm{E}[x]}{R_f} + \mathrm{Cov}(m,x)$$

Excess return of a risky asset

$$E[R^i] - R_f = -R_f \,\mathrm{Cov}(m, R^i)$$

Projection

$$x = \mathrm{proj}_m x + \varepsilon, \qquad \mathrm{proj}_m x = \frac{\mathrm{E}[mx]}{\mathrm{E}[m^2]} m, \qquad \mathrm{Cov}(m,\varepsilon) = 0.$$

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Factor Risk Models

SDF = linear combination of factors

$$m_{t+1} = a + b^T f_{t+1}$$

CAPM by Sharpe (1964)

One factor: $m_{t+1} = a + bR^w_{t+1}$, where R^w – wealth portfolio return.

Arbitrage Pricing Theory (APT) by Ross (1976)

$$R^i = a^i + b^i_1 f_1 + b^i_2 f_2 + \ldots + b^i_K f_K + \delta^i$$

$$\mathrm{E}[\delta^i] = 0$$

$$\mathrm{Cov}(\delta^i, f_k) = 0$$

Contemporary Risk Models for a Portfolio of N Stocks

Stock excess returns = factor returns + specific returns

$$r^i = \sum_{k=1}^{K} A_{ik} f_k + \delta_i.$$

- $K \ll N$;
- $\mathrm{Cov}(f_k, \delta_j) = 0$ for all k, j.
- $\mathrm{Cov}(u_i, \delta_j) = 0$ for all i, j.

Portfolio Π

$$r^{\Pi} = \sum_{i=1}^{N} w_i r_i = \sum_{k=1}^{K} A_k^{\Pi} f_k + \sum_{i=1}^{N} w_i \delta_i, \qquad A_k^{\Pi} = \sum_{i=1}^{N} w_i A_{ki}.$$

$$\mathrm{Var}\left(r^{\Pi}\right) = \sum_{k,l=1}^{K} A_k^{\Pi} A_l^{\Pi} \,\mathrm{Cov}(f_k, f_l) + \sum_{i=1}^{N} w_i^2 \,\mathrm{Var}(\delta_i)$$

Portfolio Risk

Vector Notations

N	number of stock instruments if the universe
K	number of factors
B	$K \times K$ factor covariance matrix, $B_{ij} = \mathrm{Cov}(f_i, f_j)$
A	$K \times N$ matrix of factor loadings
D	$N \times N$ diagonal matrix with entries $D_{ii} = \mathrm{Var}(u^i)$
Σ	$N \times N$ covariance matrix of the asset returns
w	vector of holdings for a portfolio Π

$$\Sigma = A^T B A + D$$

$$\mathrm{Var}\left(r^\Pi\right) = w^T \Sigma w, \qquad \sigma\left(r^\Pi\right) = \sqrt{w^T \Sigma w}.$$

Square Root of the Asset Correlation Matrix

Cholesky factorization

A symmetric positive definite matrix B can be factored:

$$B = U^{\mathsf{T}} U,$$

where U is an upper-triangular matrix.

$$\Sigma = \begin{bmatrix} I & A^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} D & \\ & B \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix}$$

Let

$$G = \begin{bmatrix} D^{1/2} & \\ & U \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} = \begin{bmatrix} D^{1/2} \\ UA \end{bmatrix} \quad \Longrightarrow \quad \Sigma = G^{\mathsf{T}} G \quad \Longrightarrow$$

$$\left(\sigma^{\Pi}\right)^2 = w^{\mathsf{T}} G^{\mathsf{T}} G w \quad \Longrightarrow \quad \sigma^{\Pi} = \| G w \|_2,$$

Portfolio Selection Optimization Problems

Notations

α	vector of stock-level return signals for a given horizon
λ	portfolio-wide risk-aversion factor

Problems

1. Minimize risk for a given level of return.
$$\mathcal{U}(w) = \sigma(w) \longrightarrow \text{min, s.t. } \alpha^T w \geq L_1$$
2. Maximize return for a given level of risk.
$$\mathcal{U}(w) = \alpha^T w \longrightarrow \text{max, s.t. } \sigma(w) \leq L_2$$
3. Markowitz mean–variance optimization.
$$\mathcal{U}(w) = -\alpha^T w + \lambda\sigma(w) \longrightarrow \text{min, s.t. } e^T w = 1.$$

Portfolio Managing Optimization Problem

Notations

w	vector of portfolio holdings
$x = \Delta w$	portfolio change (skew) vector
$TC(x)$	estimated transaction costs as function of Δw
H	sector index loadings map

Modified Markowitz optimization with TC

$$\mathcal{U}(x) = \lambda\sigma(w+x) - \alpha^{\mathsf{T}}(w+x) + TC(x) \longrightarrow \min$$

Constraints

(C1)	$m_i^- \leq w_i + x_i < m_i^+$	single name position limit
(C2)	$\sum \lvert w_i + x_i \rvert < G$	gross exposure limit
(C3)	$\sum \lvert x_i \rvert < L$	turnover limit
(C4a)	$h^- \leq H(w+x) \leq h^+$	sector net limits
(C4b)	$\left[H(w+x)\right]^{\pm} \leq g^{\pm}$	sector long/short limits

Conic Optimization with CVXOPT

Linear Conic Problem

$$
\begin{aligned}
c^T x &\longrightarrow \min \\
Gx + s &= h \\
Ax &= b \\
s &\in \mathcal{C} = C_0 \times C_1 \times \ldots \times C_M \times \ldots \times C_{M+N}
\end{aligned}
$$

Quadratic Conic Problem

$$
\begin{aligned}
\frac{1}{2} x^T P x + q^T x &\longrightarrow \min \\
Gx + s &= h \\
Ax &= b \\
s &\in \mathcal{C} = C_0 \times C_1 \times \ldots \times C_M \times \ldots \times C_{M+N}
\end{aligned}
$$

Second Order Conic Program with CVXOPT

Primal Problem

$$
\begin{aligned}
c^T x &\longrightarrow \min \\
G_0 x + s_0 &= h_0, \qquad s_0 \geq 0 \\
G_j x + s_j &= h_j, \qquad s_j = (s_j^0, s_j^1), \quad s_j^0 \geq \|s_j^1\|_2, \quad j = 1, \dots, q \\
Ax &= b
\end{aligned}
$$

Dual Problem

$$
\begin{aligned}
-\sum_{j=0}^{q} h_j^T z_j - b^T y &\longrightarrow \max \\
\sum_{j=0}^{q} G_j^T z_j + A^T y + c = 0, & \\
& z_0 \geq 0 \\
& z_j = (z_j^0, z_j^1), \quad z_j^0 \geq \|s_j^1\|_2, \quad j = 1, \dots, q
\end{aligned}
$$

SOCP formulation of Porfolio Selection Problem

1. Minimize risk for a given level of return.

$\mathcal{U}(w) = \sigma(w) \longrightarrow \min$, s.t. $\alpha^T w \geq L_1$, $e^T w = 1$.

SOCP formulation of Porfolio Selection Problem

2. Maximize return for a given level of risk.

$\mathcal{U}(w) = \alpha^T w \longrightarrow \max$, s.t. $\sigma(w) \leq L_2$, $e^T w = 1$.

SOCP formulation of Porfolio Selection Problem

3. Markowitz mean–variance optimization.

$\mathcal{U}(w) = -\alpha^T w + \lambda\sigma(w) \longrightarrow \min$, s.t. $e^T w = 1$.

References

Books

- *Asset Pricing* by John H. Cochrane
- *Robust Portfolio Optimization and Management* by Frank J. Fabozzi and others.
- *Lectures on Modern Convex Optimization* by Aharon Ben-Tal and Arkadi Nemirovski.

Links

- http://cvxopt.org
- http://docs.mosek.com/whitepapers/portfolio.pdf